# LEFT DETERMINED MORPHISMS AND FREE REALISATIONS

LORNA GREGORY

*This paper is dedicated to Mike Prest on the occasion of his 65th birthday.*

Abstract. We investigate the connection between Prest's notion of the free realisation of a pp formula and Auslander's notion of determiners of functor and morphisms.

The aim of this note is to explain the connections between Auslander's notion of morphisms and subfunctors determined by objects introduced in [Aus78] and Prest's notion of free realisations of pp formulae introduced in [Pre88].

The concept of determiners of morphisms and subfunctors were largely ignored until recently. In the last 5-10 years, effort has been made to understand them (see for instance [Rin13], [Rin12], [Kra13]). On the other hand, the algebraic study of model theory of modules is unimaginable without the concept of free realisations of a pp formulae.

In 2.4 we explicitly describe the connection between determiners of functors defined by pp formulae and free realisations of pp formulae. This will give another proof, 2.5, of the existence of left determiners of morphisms between finitely presented modules for artin algebras. We then use determiners and free realisations to show that if $M \in \text{mod-}R$ and $R$ is an artin algebra, then the lattice homomorphism $\text{pp}_R^k \to \text{pp}_R^k(M)$ which sends $\varphi \in \text{pp}_R^k$ to $\varphi(M) \in \text{pp}_R^k(M)$ has both a left and a right adjoint, both of which we explicitly describe.

Finally, in section 3, we will show that pushing the ideas from section 2 slightly harder actually gives a proof of the existence of minimal left determiners of morphisms between finitely presented modules for artin algebras.

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## 1. Basic concepts

The material in this section about morphisms and functors determined by objects is from the introduction of [Aus78]. The material about pp formulae and free realisations can be found in [Pre88] and [Pre09].

If $R$ is a ring then we write Mod-$R$ (resp. $R$-Mod) for the category of right (resp. left) $R$-modules and mod-$R$ (resp. $R$-mod) for the category of finitely presented right (resp. left) $R$-modules. If $\mathcal{C}$ is an additive category then $(\mathcal{C}, \mathrm{Ab})$ will denote the category with objects the additive functors and morphisms natural transformations.

**Definition 1.1.** Let $\mathcal{C}$ be an additive category and $G \in (\mathcal{C}, \mathrm{Ab})$ (resp. $G \in (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$). We say that a subfunctor $H \subseteq G$ is **determined** by $C \in \mathcal{C}$ if for all subfunctors $H' \subseteq G$, $H' \subseteq H$ if and only if $H'C \subseteq HC$. We will call $C$ a **determiner** for $H$.

We will say that $C$ is a **minimal determiner** for $H$ as a subfunctor of $G$, if $C$ is a direct summand of all other determiners for $H$.

**Definition 1.2.** Let $f : Y \to X$ be a morphism in an additive category $\mathcal{C}$.

(1) We say that $f$ is **right determined** by $C \in \mathcal{C}$ provided that, for all $\beta : Z \to X$, if for all $\gamma : C \to Z$, $\beta\gamma$ factors through $f$ then $\beta$ factors through $f$.

(2) We say that $f$ is **left determined** by $C \in \mathcal{C}$ provided that, for all $\beta : Y \to Z$, if for all $\gamma : Z \to C$, $\gamma\beta$ factors through $f$ then $\beta$ factors through $f$.

We will refer to $C$ as a right (resp. left) **determiner** for $f$ and say that $C$ is a **minimal right (resp. left) determiner** for $f$ if $C$ is a direct summand of all other right (resp. left) determiner for $f$.

**Lemma 1.3.** *Let $\mathcal{C}$ be an additive category.*

*(1) A morphism $f : X \to Y$ is right determined by $C$ if and only if $\mathrm{im}(-, f)$ is determined by $C$ as a subfunctor of $(-, Y)$.*

*(2) A morphism $f : Y \to X$ is left determined by $C$ if and only if $\mathrm{im}(f, -)$ is determined by $C$ as a subfunctor of $(Y, -)$.*

*Remark* 1.4. A morphism $f : X \to Y \in \mathcal{C}$ is right determined by $C \in \mathcal{C}$ if and only if $f^{\mathrm{op}} : Y \to X \in \mathcal{C}^{\mathrm{op}}$ is left determined by $C \in \mathcal{C}^{\mathrm{op}}$.

Let $R$ be an Artin algebra over a commutative Artinian ring $S$. The injective envelope $I$ of $S/\mathrm{rad}(S)$ is a finitely presented injective cogenerator for Mod-$S$. If $M \in$ mod-$R$ (resp. $M \in R$-mod) then $M^* := \mathrm{Hom}_S(M, I)$ is finitely presented as a left (resp. right) $R$-module and if $f : M \to N \in$ mod-$R$ (resp. $f : M \to N \in R$-mod) then write $f^* := \mathrm{Hom}_S(f, I)$. The contravariant functor $\mathrm{Hom}_S(-, I) :$ mod-$R \to R$-mod gives an equivalence of categories $(\text{mod-}R)^{op} \simeq R\text{-mod}$.

Specialising to the case where $\mathcal{C} := \text{mod-}R$ (resp. $\mathcal{C} := R\text{-mod}$) and $R$ is an Artin algebra, 1.4 shows that for $A, B \in \text{mod-}R$, $f : A \to B$ is right determined by $X$ if and only $f^* : B^* \to A^*$ is left determined by $X^*$.

Auslander showed in [Aus78], if $R$ is an Artin algebra, then all morphisms $f : M \to N \in \text{mod-}R$ have minimal right and left determiners.

**Definition 1.5.** Let $R$ be a ring. A (right) **pp-$n$-formula** is a formula in the language $\mathcal{L}_R = (0, +, (\cdot r)_{r\in R})$ of (right) $R$-modules of the form

$$\exists \overline{y}(\overline{x}, \overline{y})H = 0$$

where $\overline{x}$ is a $n$-tuple of variables and $H$ is an appropriately sized matrix with entries in $R$.

If $\varphi$ is a pp-$n$-formula and $M$ is a module then we will write $\varphi(M)$ for the solution set of $\varphi$ in $M^n$. Solutions sets of pp formulae have two important (and easily checked) properties. Firstly, the solution set of a pp-$n$-formula $\varphi$ in a module $M$ is a subgroup of $M^n$ under pointwise addition. Secondly, if $f : M \to N$ is a homomorphism of $R$-modules then the image of $\varphi(M)$ under $f$ is contained in $\varphi(N)$. In this way, pp formulae give rise to functors in $(\text{mod-}R, \text{Ab})$ i.e. if $\varphi$ is a pp formula then we define a functor $F_\varphi \in (\text{mod-}R, \text{Ab})$ which acts on objects by sending $M \in \text{mod-}R$ to $\varphi(M)$ and morphisms in the obvious way. If $\varphi$ is a pp-$n$-formula, then $F_\varphi$ is a subfunctor of the $n$th power of the forgetful functor $(R^n, -)$.

These properties also imply that if $M \in \text{Mod-}R$ and $\varphi$ is a pp-$n$-formula then $\varphi(M)$ is closed under the diagonal action of $\text{End}(M)$ on $M^n$.

Note also that solution sets of pp formulae commute with direct sums. That is, if $\varphi \in \text{pp}_R^n$ and $L, M \in \text{Mod-}R$ then $\varphi(L \oplus M) = \varphi(L) \oplus \varphi(M)$.

After identifying pp-$n$-formulae $\varphi, \psi$ such that $\varphi(M) = \psi(M)$ for all $M \in \text{Mod-}R$, the set of (equivalence classes of) pp-$n$-formulae becomes a lattice by setting $\varphi \leq \psi$ if and only if $\varphi(M) \subseteq \psi(M)$ for all $M \in \text{Mod-}R$. We will write $\text{pp}_R^n$ for the lattice of right pp-$n$-formulae and ${}_R\text{pp}^n$ for the lattice of left pp-$n$-formulae. If $M \in \text{Mod-}R$ (resp. $M \in R\text{-Mod}$) then we will write $\text{pp}_R^n(M)$ (resp. ${}_R\text{pp}^n(M)$) for the lattice of all subgroups of $M^n$ defined by right (resp. left) pp-$n$-formulae. This is just the quotient of $\text{pp}_R^n$ (resp. ${}_R\text{pp}^n$) by the equivalence relation $\varphi \sim \psi$ if $\varphi(M) = \psi(M)$.

If $R$ is an Artin algebra over a commutative Artinian ring $S$ and $M$ is a finitely presented $R$-module then all left $\text{End}(M)$-submodules of $M^n$ are also $S$-submodules. Since $M$ is a finite length as $S$-module, $M^n$ is finite length as a left $\text{End}(M)$-module and hence $\text{pp}_R^n(M)$ is finite length.

If $\overline{m}$ is an $n$-tuple of elements from a module $M$ then the **pp-type** of $\overline{m}$ is simply the set of pp-$n$-formulae $\varphi$ such that $\overline{m} \in \varphi(M)$. If $M \in \text{mod-}R$ and $\overline{m}$ is an $n$-tuple of element from $M$ then, [Pre09, 1.2.6], there exists $\varphi \in \text{pp}_R^n$ such that $\psi$ is in the pp-type of $\overline{m}$ if and only if $\psi \geq \varphi$.

**Definition 1.6.** Let $\varphi \in \text{pp}_R^n$. A **free realisation** of $\varphi$ is a pair $(C, \overline{c})$ where $C \in \text{mod-}R$ and $\overline{c}$ is an $n$-tuple of elements from $C$ with the property that the pp-type of $\overline{c}$ in $C$ is generated by $\varphi$ i.e. $\psi(\overline{c})$ holds in $C$ if and only if $\varphi \leq \psi$.

**Proposition 1.7.** [Pre09, 1.2.14,1.2.7] *Every pp formula $\varphi$ has a free realisation. Moreover, if $(C, \overline{c})$ is a free realisation for $\varphi$ and $\overline{m} \in \varphi(M)$ for some module $M$ and tuple $\overline{m}$ of elements from $M$ then there is a homomorphism $f : C \to M$ such that $f(\overline{c}) = \overline{m}$.*

If $\overline{c} \in C^n$ then we write $\overline{c} : R^n \to C$ for the map given by $\overline{r} \mapsto \overline{c} \cdot \overline{r}$. Equivalently, see [Pre09, 10.2.8], a free realisation is a pair $(C, \overline{c})$ where $C \in \text{mod-}R$ and $\overline{c}$ is an $n$-tuple of elements from $C$ with the property that $\text{im}(\overline{c}, -) = F_\varphi$. This in fact implies that $F_\varphi$ is finitely presented and that all finitely presented subfunctors of $(R^n, -)$ are of the form $F_\varphi$ for some $\varphi \in \text{pp}_R^n$.

A free realisation $(C, \overline{c})$ of a pp formula $\varphi$ is **minimal** if whenever $f \in \text{End}(C)$ and the pp-type of $\overline{c}$ and $f(\overline{c})$ are equal, $f$ is an automorphism. Equivalently, $(\overline{c}, -) : (C, -) \to F_\varphi$ is a projective cover. Over an Artin algebra, or in fact over any Krull-Schmidt ring, all pp formulae have minimal free realisations [Pre09, 4.3.70].

**Lemma 1.8.** [Pre09, 10.2.26] *Let $(C, \overline{c})$ be a minimal free realisation for a pp formula $\varphi$. If $(A, \overline{a})$ is a free realisation for $\varphi$ then there is a split monomorphism such $g : C \to A$ such that $g(\overline{c}) = \overline{a}$.*

For each $n \in \mathbb{N}$, Prest defined a lattice anti-isomorphism $D : \text{pp}_R^n \to {}_R\text{pp}^n$ (see [Pre09, section 1.3.1] and [Pre88, 8.21]). As is standard, we denote its inverse ${}_R\text{pp}^n \to \text{pp}_R^n$ also by $D$. Since we will not need to explicitly take the dual of a pp formula here, we will not give its definition.

**Theorem 1.9.** [Her93, 3.2][Pre09, 1.3.7] *Let $\overline{a}$ and $\overline{b}$ be $n$-tuples from $M \in \text{Mod-}R$ and $N \in R\text{-Mod}$ respectively. Then $\overline{a} \otimes \overline{b} = 0$ if and only if there exists $\varphi \in \text{pp}_R^n$ such that $\overline{a} \in \varphi(M)$ and $\overline{b} \in D\varphi(N)$.*

**Corollary 1.10.** *If $(C, \overline{c})$ is a free realisation of $\varphi \in \text{pp}_R^n$ and $\overline{l}$ is an $n$-tuple for $L \in R\text{-Mod}$ then $\overline{c} \otimes \overline{l} = 0$ if and only if $\overline{l} \in D\varphi(L)$.*

**Lemma 1.11.** [Pre09, 1.3.13] *Let $R$ be an Artin algebra, $M \in \text{Mod-}R$ and $\varphi, \psi \in \text{pp}_R^n$. If $\psi(M) \subseteq \varphi(M)$ then $D\varphi(M^*) \subseteq D\psi(M^*)$.*

## 2. The relationship between free realisations and determiners

In this section we give a correspondence between determiners of pp formulae as subfunctors of $(R^n, -)$ and modules which free realise their dual. We then use free realisations and determiners to define a canonical meet-semi-lattice embedding $\rho_M : \text{pp}_R^k(M) \to \text{pp}_R^k$ and a canonical join-semi-lattice embedding $\lambda_M : \text{pp}_R^k(M) \to \text{pp}_R^k$ whenever $R$ is an Artin algebra and $M \in \text{mod-}R$.

Suppose that $\varphi \in \text{pp}_R^k$ and $(M, \overline{m})$ is a free realisation for $\varphi$. Then $\varphi(M) \subseteq \psi(M)$ implies that $\overline{m} \in \psi(M)$. Hence $\varphi \leq \psi$. So $\varphi \leq \psi$ if and only if $\varphi(M) \subseteq \psi(M)$. The following lemma gives a partial converse to this.

**Lemma 2.1.** *Let $M \in \text{mod-}R$ be such that $\text{pp}^1_R(M)$ has the ascending chain condition. If $\varphi \in \text{pp}^k_R$ is such that for all $\psi \in \text{pp}^k_R$, $\psi(M) \supseteq \varphi(M)$ implies $\psi \geq \varphi$, then there exist $n \in \mathbb{N}$ and $\overline{m} \in M^n$ such that $\varphi$ is freely realised by $(M^n, \overline{m})$.*

*Proof.* Suppose that $\varphi$ and $M$ are as in the statement of this lemma. We will define a finite sequence, indexed by $i$, of $k$-tuples $\overline{m}_i$ of elements from $M^i$ such that $\overline{m}_i \in \varphi(M^i)$ and if $\chi_i$ generates the pp-type of $\overline{m}_i$ then $\chi_{i+1}(M) \supsetneq \chi_i(M)$. Now, since $\text{pp}^1_R(M)$ has the ascending chain condition, this process will terminate with $\overline{m}_n$ a $k$-tuple of elements from $M^n$ and by the details of the construction below, $\overline{m}_n$ will be such that $\varphi(M) = \chi_n(M)$. So, by hypothesis and since $\varphi(M) = \chi_n(M)$, $\chi_n \geq \varphi$. Since $\overline{m}_n \in \varphi(M^n)$ and $\chi_n$ generates the pp-type of $\overline{m}_n$, we have that $\varphi \geq \chi_n$. Thus $\varphi = \chi_n$ and $(M^n, \overline{m}_n)$ freely realises $\varphi$.

Pick $\overline{m}_1 \in \varphi(M)$. If $\chi_1(M) = \varphi(M)$ then we are done. Supposing that we have already defined $\overline{m}_i \in M^i$ and $\chi_i(M) \subsetneq \varphi(M)$, pick $\overline{m}' \in \varphi(M)\backslash\chi_i(M)$. Let $\overline{m}_{i+1} = \overline{m}_i \oplus \overline{m}' \in M^{i+1}$. Since the solution sets of pp-formulae commute with direct sums, for all $\psi \in \text{pp}^k_R$, $\overline{m}_{i+1} \in \psi(M^{i+1})$ if and only if $\overline{m}_i \in \psi(M^i)$ and $\overline{m}' \in \psi(M)$. Thus for all $\psi \in \text{pp}^k_R$, $\overline{m}_{i+1} \in \psi(M^{i+1})$ implies $\psi \geq \chi_i$. Thus $\chi_{i+1} \geq \chi_i$. Since $\overline{m}' \in \chi_{i+1}(M)$ and $\overline{m}' \notin \chi_i(M)$, we have that $\chi_{i+1}(M) \supsetneq \chi_i(M)$ as required. □

If $M \in \text{Mod-}R$ is length $e$ as a left $\text{End}(M)$-module then $M^k$ is length $ek$ as a left $\text{End}(M)$-module. Since for all $\varphi \in \text{pp}^k_R$, $\varphi(M)$ is a left $\text{End}(M)$-submodule of $M^k$, $\text{pp}^k_R(M)$ is at most length $ek$. Thus we have the following corollary to the proof of 2.1.

**Corollary 2.2.** *Let $M \in \text{mod-}R$ be of length $e$ as a left $\text{End}(M)$-module. If $\varphi \in \text{pp}^k_R$ is such that for all $\psi \in \text{pp}^k_R$, $\psi(M) \supseteq \varphi(M)$ implies $\psi \geq \varphi$, then there exists $\overline{m} \in M^{ek}$ such that $\varphi$ is freely realised by $(M^{ek}, \overline{m})$.*

**Corollary 2.3.** *Let $R$ be an Artin algebra. A pp-$k$-formula $\varphi$ over $R$ is freely realised in $M^n$ for some $n \in \mathbb{N}$ if and only if $\varphi$ is freely realised in $M^{ek}$ where $e$ is the length of $M$ as a left $\text{End}(M)$-module.*

**Theorem 2.4.** *Let $R$ be an Artin algebra, $\varphi$ a pp-$k$-formula and $M \in \text{mod-}R$. The following are equivalent:*

1. *$M^*$ determines $F_{D\varphi}$ as a subfunctor of $(R^k, -)$*
2. *there is some $n \in \mathbb{N}$ and $k$-tuple $\overline{m}$ of elements from $M^n$ such that $(M^n, \overline{m})$ freely realises $\varphi$.*

*Proof.* (2)⇒ (1): Suppose that $\varphi \in \text{pp}^k_R$ and $(M^n, \overline{m})$ is a free realisation of $\varphi$. If $\psi \in \text{pp}^k_R$ then, since $(M^n, \overline{m})$ is a free realisation of $\varphi$, $\varphi \leq \psi$ if and only if $\overline{m} \in \psi(M^n)$. So $\varphi(M^n) \subseteq \psi(M^n)$ implies $\overline{m} \in \psi(M^n)$ and hence $\varphi \leq \psi$. Thus $\varphi \leq \psi$ if and only if $\varphi(M^n) \subseteq \psi(M^n)$ if and only if $\varphi(M) \subseteq \psi(M)$.

Taking duals, this gives that for all $\psi \in \text{pp}^n_R$, $D\psi \leq D\varphi$ if and only if $D\psi(M^*) \subseteq D\varphi(M^*)$. Since, see [Pre09, 12.2.1], all subfunctors of $(R^n, -) \in$

$(R\text{-mod}, \mathrm{Ab})$ are direct unions of functors of the form $F_{D\psi}$, this implies that for all $G \subseteq (R^n, -)$, $G \subseteq F_{D\varphi}$ if and only if $GM^* \subseteq F_{D\varphi}M^*$. That is $F_{D\varphi}$ is determined, as a subfunctor of $(R^n, -)$, by $M^*$.
(1)$\Rightarrow$ (2) Since $M^*$ is a determiner for $F_{D\varphi}$, we have that for all $\psi \in \mathrm{pp}_R^k$, $D\psi(M^*) \subseteq D\varphi(M^*)$ if and only if $D\psi \leq D\varphi$. So $\psi(M) \supseteq \varphi(M)$ if and only if $\psi \geq \varphi$. Since $M$ is a finitely presented module over an Artin algebra, it is of finite length as a left $\mathrm{End}(M)$-module. Thus, 2.1 implies there exists an $n \in \mathbb{N}$ and $k$-tuple $\overline{m}$ of elements from $M^n$ such that $(M^n, \overline{m})$ freely realises $\varphi$.

□

The following corollary explains how the existence of free realisations of pp formulae implies the existence of left determiners for morphisms between finitely presented modules over an Artin algebra.

**Corollary 2.5.** *Let $R$ be an Artin algebra, $f : A \to B \in \text{mod-}R$ and $\overline{a}$ a generating tuple for $A$. Let $\varphi$ generate the pp-type of $f(\overline{a})$. If some tuple of elements in $C \in R\text{-mod}$ freely realises $D\varphi$ then $C^*$ is a left determiner for $f$.*

*Proof.* Suppose that $\overline{a}$ is of length $k$ and let $g : R^k \to A$ be the map defined by $g(\overline{r}) := \overline{a} \cdot \overline{r}$. Since $F_\varphi = \mathrm{im}(f \circ g, -)$, $C^*$ is a determiner for $\mathrm{im}(f \circ g, -)$ as a subfunctor of $(R^n, -)$. Since $g$ is an epimorphism, $(g, -) : (A, -) \to (R^n, -)$ is a monomorphism. Thus $C^*$ is also a determiner for $\mathrm{im}(f, -)$ as a subfunctor of $(A, -)$. □

We now discuss the relationship between minimal free realisations of $\varphi \in \mathrm{pp}_R^k$ and minimal determiners of $F_{D\varphi}$. Suppose that $(C, \overline{c})$ is a minimal free realisation for $\varphi$ and $D$ is a minimal determiner of $F_{D\varphi}$ as a subfunctor of $(R^k, -)$. Then, by 1.8 and 2.4, $C$ is a direct summand of $(D^*)^n$ for some $n \in \mathbb{N}$ and $D$ is a direct summand of $C^*$. Thus $C^*$ and $D$ have the same indecomposable direct summands. Thus if $(C, \overline{c})$ is a minimal free realisation for $\varphi$ and $C = C_1^{l_1} \oplus \ldots \oplus C_m^{l_m}$, where $C_i \cong C_j$ implies $i = j$, then $(C_1 \oplus \ldots \oplus C_m)^*$ is a minimal determiner for $F_{D\varphi}$.

The following lemma indicates how far 2.5 combined with minimal free realisations is from giving us minimal left determiners.

**Lemma 2.6.** *Let $R$ be an Artin algebra. Let $f : A \to B \in \text{mod-}R$ and $g : R^n \twoheadrightarrow A$. If $C \in \text{mod-}R$ left determines $f$ then $C \oplus I$ left determines $g \circ f$ where $I$ is an injective cogenerator for $\text{mod-}R$.*

*Proof.* First note that $g : R^n \twoheadrightarrow A$ is left determined by $I$. One way to see this is that $\mathrm{im}(g, -)$ is equivalent to $F_\varphi$ for some quantifier free formula $\varphi$ [Pre09, 1.2.30] and thus its dual $D\varphi$ is freely realised in a projective module [Pre09, 1.2.29] and the dual of a projective is injective.

Now if $F \subseteq (R^n, -)$ and $F(C \oplus I) \subseteq \mathrm{im}(g \circ f, C \oplus I)$ then $FI \subseteq \mathrm{im}(g, I)$. Thus $F \subseteq \mathrm{im}(g, -)$. So $F$ is a subfunctor of $\mathrm{im}(g, -)$. Let $F' \subseteq (A, -)$ be the inverse image of $F$ under $(g, -)$. Now $F(C \oplus I) \subseteq \mathrm{im}(g \circ f, C \oplus I)$

implies $F'(C) \subseteq \mathrm{im}(f, C)$ since $(g, -)$ and hence $(g, C)$ is an embedding. So $F' \subseteq \mathrm{im}(f, -)$. So $F \subseteq \mathrm{im}(g \circ f, -)$. □

We now finish this section by defining two canonical poset embeddings of the lattice of pp-definable subsets of finite length modules over an Artin algebra with the first a meet semi-lattice embedding and the second a join semi-lattice embedding.

If $\varphi \in \mathrm{pp}^k_R$ and $M \in \text{mod-}R$, then we will now write $M$ determines $\varphi$ to mean that $M$ determines $F_\varphi$ as a subfunctor of $(R^k, -)$.

**Lemma 2.7.** *Let $R$ be an Artin algebra and $M \in \text{mod-}R$. For any $\chi \in \mathrm{pp}^k_R$ there exist unique $\varphi, \psi \in \mathrm{pp}^k_R$ such that $\varphi(M) = \chi(M) = \psi(M)$, $\varphi$ is freely realised in $M^n$ for some $n \in \mathbb{N}$ and $F_\psi$ is determined by $M$.*

*Proof.* We want to show that there is a $\varphi \in \mathrm{pp}^k_R$ such that $\varphi(M) = \chi(M)$ and $\varphi$ is freely realised in $M^n$ for some $n \in \mathbb{N}$. We will define a finite sequence, indexed by $i$, of $k$-tuples $\overline{m}_i$ of elements from $M^i$ such that $\overline{m}_i \in \chi(M^i)$ and if $\varphi_i$ generates the pp-type of $\overline{m}_i$ then $\varphi_{i+1}(M) \supsetneq \varphi_i(M)$. Since $M$ is finite endolength this process must terminate with $\overline{m}_n \in M^n$ and by details of the construction below, $\overline{m}_n$ will be such that $\varphi_n(M) = \chi(M)$.

Pick $\overline{m}_1 \in \chi(M)$. If $\chi(M) = \varphi_1(M)$ then we are done. Suppose that we have already defined $\overline{m}_i \in M^i$ and that $\chi(M) \supsetneq \varphi_i(M)$. Pick $\overline{m}' \in \chi(M) \backslash \varphi_i(M)$. Let $\overline{m}_{i+1} = m_i \oplus m'$. We have that $\overline{m}_{i+1} \in \chi(M)$ and that $\varphi_{i+1} \geq \varphi_i$ since $\overline{m}_i \in \varphi_{i+1}(M)$. Hence, since $m' \notin \varphi_i(M)$ and $m' \in \varphi_{i+1}(M)$, $\varphi_{i+1}(M) \supsetneq \varphi_i(M)$.

Now suppose that $\varphi_1$ and $\varphi_2$ are both freely realised in some power of $M$ and that $\chi(M) = \varphi_1(M) = \varphi_2(M)$. Then $M^*$ determines $D\varphi_1$ and $D\varphi_2$. Moreover $D\varphi_1(M^*) = D\varphi_2(M^*) = D\chi(M^*)$, so $D\varphi_1 = D\varphi_2$. Thus $\varphi_1 = \varphi_2$.

Let $\psi$ be such that $D\psi$ is freely realised in $(M^*)^n$ for some $n \in \mathbb{N}$ and $D\psi(M^*) = D\chi(M^*)$. Then $M$ determines $\psi$ and $\psi(M) = \chi(M)$ as required. □

Let $R$ be an Artin algebra and $M \in \text{mod-}R$. We define three order preserving maps based on the above lemma.

(1) Let $\rho_M : \mathrm{pp}^k(M) \to \mathrm{pp}^k_R$ take $\chi(M)$ to the unique $\varphi \in \mathrm{pp}^k_R$ such that $\varphi(M) = \chi(M)$ and $\varphi$ is determined by $M$.
(2) Let $\lambda_M : \mathrm{pp}^k(M) \to \mathrm{pp}^k_R$ take $\chi(M)$ to the unique $\psi \in \mathrm{pp}^k_R$ such that $\psi(M) = \chi(M)$ and $\psi$ is freely realised in $M^n$ for some $n \in \mathbb{N}$.
(3) Let $\mu_M : \mathrm{pp}^k_R \to \mathrm{pp}^k(M)$ be the map taking $\chi \in \mathrm{pp}^k_R$ to $\chi(M)$.

We will use the same notation for the left module versions of these maps. Note that if $M \in \text{mod-}R$ (resp. $M \in R\text{-mod}$), by 1.11, the duality $D : \mathrm{pp}^k_R \to {}_R\mathrm{pp}^k$ (resp. $D : {}_R\mathrm{pp}^k \to \mathrm{pp}^k_R$) induces a duality $D_M : \mathrm{pp}^k_R(M) \to {}_R\mathrm{pp}^k(M^*)$ (resp. $D_M : {}_R\mathrm{pp}^k(M) \to \mathrm{pp}^k_R(M^*)$).

**Lemma 2.8.** *Let $R$ be an Artin algebra and $M \in \text{mod-}R$.*

*(i) The functions $\rho_M, \lambda_M, \mu_M$ are order preserving.*

*(ii)* $\rho_M = D \circ \lambda_{M^*} \circ D_M$ *and* $\lambda_M = D \circ \rho_{M^*} \circ D_M$.

*(iii)* $\mu_M \circ \rho_M = Id_{\mathrm{pp}^k_R(M)}$ *and* $\mu_M \circ \lambda_M = Id_{\mathrm{pp}^k_R(M)}$

*(iv)* *For all* $\psi \in \mathrm{pp}^k_R$, $\rho_M(\psi(M))$ *is the largest element in*

$$\{\varphi \in \mathrm{pp}^k_R \mid \varphi(M) \subseteq \psi(M)\}$$

*and* $\lambda_M(\psi(M))$ *is the smallest element in*

$$\{\varphi \in \mathrm{pp}^k_R \mid \psi(M) \subseteq \varphi(M)\}.$$

*(v)* *If the ordered sets* $\mathrm{pp}^k_R$ *and* $\mathrm{pp}^k_R(M)$ *are viewed as categories in the usual way and* $\lambda_M, \mu_M$ *and* $\rho_M$ *as functors then* $\lambda_M \dashv \mu_M \dashv \rho_M$.

*(vi)* $\rho_M$ *is a meet-semi-lattice embedding and* $\lambda_M$ *is a join-semi-lattice embedding.*

*Proof.* $(i)$ Suppose that $\varphi, \psi \in \mathrm{pp}^k_R$ and $\varphi(M) \subseteq \psi(M)$. Then $\varphi(M) = \rho_M(\varphi(M))(M) \subseteq \rho_M(\psi(M))(M) = \psi(M)$ and $\rho_M(\psi(M))$ is determined by $M$. So $\rho_M(\varphi(M)) \leq \rho_M(\psi(M))$. Thus $\rho_M$ is order preserving. That $\lambda_M$ is order preserving follows similarly and $\mu_M$ is order preserving by definition of the order on $\mathrm{pp}^k_R$.
$(ii)$ We show that $D \circ \rho_M = \lambda_{M^*} \circ D_M$. From this both equalities can be deduced. Suppose $\chi \in \mathrm{pp}^k_R$. Then $\rho_M(\chi(M))$ is determined by $M$ and $\rho_M(\chi(M))(M) = \chi(M)$. So $D \circ \rho_M(\chi(M))(M^*) = D\chi(M^*)$ by 1.11 and $D \circ \rho_M(\chi(M))$ is freely realised in $(M^*)^n$ for some $n \in \mathbb{N}$ by 2.4. Thus $D \circ \rho_M(\chi(M)) = \lambda_{M^*}(D\chi(M^*)) = \lambda_{M^*} \circ D_M(\chi(M))$.
$(iv)$ By definition $\rho_M(\psi(M))(M) = \psi(M) \subseteq \psi(M)$ and since $\rho_M(\psi(M))$ is determined by $M$, $\varphi(M) \subseteq \psi(M) = \rho_M(\psi(M))(M)$ implies $\varphi \leq \rho_M(\psi(M))$.
$(v)$ It is well known and easy to check that if $g : P_1 \to P_2$ and $f : P_2 \to P_1$ are functors between preordered sets viewed as categories then $g$ is right adjoint to $f$ exactly if, for all $a \in P_2$ and $b \in P_1$, $f(a) \leq b$ if and only if $a \leq g(b)$.

Let $\varphi, \psi \in \mathrm{pp}^k_R$. If $\varphi \leq \rho_M(\psi(M))$ then $\mu_M(\varphi) \leq \psi(M)$ since $\mu_M \circ \rho_M = \mathrm{Id}_{\mathrm{pp}(M)}$. Suppose that $\mu_M(\varphi) \leq \psi(M)$ i.e. $\varphi(M) \subseteq \psi(M)$. Then $\rho_M(\psi(M))(M) = \psi(M)$ and $\rho_M(\psi(M))$ is determined by $M$. So $\varphi \leq \rho_M(\psi(M))$. Thus we have shown that $\rho_M$ is right adjoint to $\mu_M$.
$(vi)$ That $\rho_M$ and $\lambda_M$ are embeddings of partially ordered sets is implied by $(iii)$. Meets in preorders are products and joins in preorders are coproducts. So $(vi)$ follows directly from $(v)$ since right adjoints preserve limits and left adjoints preserve colimits. □

## 3. A proof of the existence of left determiners for morphisms

In this section we give a proof of the existence of minimal left determiners for morphisms in mod-$R$ when $R$ is an Artin algebra. This proof is inspired by 2.4 and we explain how the various steps correspond to those in 2.4.

As we have shown, the existence of minimal free realisations for pp formulae comes very close to implying the existence of minimal left determiners for morphisms in mod-$R$ where $R$ is an Artin algebra.

Let $R$ be a ring and $\varphi$ is a pp-$n$-formula. Suppose $f : R^n \to M \in \text{mod-}R$ and $\text{im}(f,-) = F_\varphi \subseteq (R^n,-)$. Then, by 1.10,

$$0 \longrightarrow F_{D\varphi} \longrightarrow R^n \otimes - \cong (R^n,-) \xrightarrow{f\otimes -} M \otimes -$$

is exact. A free realisation for $D\varphi(-)$ is just a map $\gamma : R^n \to N \in R\text{-mod}$ such that $(\gamma,-) : (N,-) \to (R^n,-)$ has image $F_{D\varphi(-)}$ and a free realisation is minimal if $(\gamma,-) : (N,-) \to (R^n,-)$ is a projective cover for its image $F_{D\varphi(-)}$.

**Theorem 3.1.** *Let $R$ be an Artin algebra. If $\gamma : (D,-) \to \ker(f \otimes -)$ is a projective cover then $C \in \text{mod-}R$ is a left determiner for $f$ if and only if $D^*$ is a direct summand of $C^n$ for some $n \in \mathbb{N}$. That is, if $D = D_1^{l_1} \oplus \ldots \oplus D_m^{l_m}$, where $D_i \cong D_j$ implies $i = j$, then $(D_1 \oplus \ldots \oplus D_m)^*$ is a minimal left determiner for $f$.*

This statement follows from [Kra13, 3.13]. However, our proofs are somewhat more elementary and our motivation for the statement and proof have different roots.

In order to prove the above theorem we first prove 4 lemmas.

The first generalises elementary duality for pp formulae. The functor $\text{im}(g,-)$ plays the role of a pp formula $\varphi$ and the functor $\ker(g \otimes -)$ plays the role of $D\varphi$.

**Lemma 3.2.** *Let $R$ be a ring. Let $f : A \to B$ and $g : A \to C$ be morphisms in* mod-$R$. *The following are equivalent:*

*(1)* $\text{im}(g,-) \subseteq \text{im}(f,-)$
*(2) there exists $h : B \to C$ such that $g = h \circ f$*
*(3)* $\ker(f \otimes -) \subseteq \ker(g \otimes -)$

*Proof.* (1)⇒(2) Since $\text{im}(g,C) \subseteq \text{im}(f,C)$, $g \in \text{im}(f,C)$ and hence there exists $h : B \to C$ such that $g = h \circ f$.

(2)⇒(3) We have that $g \otimes - = (h \otimes -) \circ (f \otimes -)$.

(3)⇒(1) Let $d$ be the Auslander-Gruson-Jensen dual as in [Pre09, section 10.3]. Since $d$ is exact and $d(g \otimes -) = (g,-)$, the cokernel of $(f,-)$ factors through the cokernel of $(g,-)$. So $\text{im}(g,-) \subseteq \text{im}(f,-)$. □

The next lemma generalises the statement that if $R$ is an Artin algebra, $\varphi, \psi \in \text{pp}_R^k$ and $M \in \text{mod-}R$ then $\varphi(M) \supseteq \psi(M)$ if and only if $D\psi(M^*) \supseteq D\varphi(M^*)$.

**Lemma 3.3.** *Suppose that $R$ is an Artin algebra. Let $f : A \to B$, $g : A \to C$ be morphisms in* mod-$R$ *and let $D \in$* mod-$R$. *The following are equivalent:*

*(1)* $\text{im}(g,D) \subseteq \text{im}(f,D)$
*(2)* $\ker(f \otimes D^*) \subseteq \ker(g \otimes D^*)$

*Proof.* Throughout, let $\overline{a}$ be a tuple of elements from $A$, $\overline{t} \in D^*$, $\psi$ generate the pp-type of $f(\overline{a})$ and $\varphi$ generate the pp-type of $g(\overline{a})$. Thus, by 1.10,

$f(\overline{a}) \otimes \overline{t} = 0$ if and only if $\overline{t} \in D\psi(D^*)$ and $g(\overline{a}) \otimes \overline{t} = 0$ if and only if $\overline{t} \in D\varphi(D^*)$.
(1)⇒(2) Since $\mathrm{im}(g, D) \subseteq \mathrm{im}(f, D)$,

$$\varphi(D) = \{\delta g\overline{a} \mid \delta \in (C, D)\} \subseteq \{\gamma f\overline{a} \mid \gamma \in (B, D)\} = \psi(D).$$

Therefore, by 1.11, $D\psi(D^*) \subseteq D\varphi(D^*)$. Suppose that $\overline{a}\otimes\overline{t} \in \ker(f \otimes D^*) \subseteq A \otimes D^*$. Then $f(\overline{a}) \otimes t = 0$. So $\overline{t} \in D\psi(D^*)$. So $\overline{t} \in D\varphi(D^*)$. Thus $g(\overline{a}) \otimes \overline{t} = 0$, that is $\overline{a} \otimes \overline{t} \in \ker(g \otimes D^*)$.
(2)⇒(1) Let $\overline{a}$ generate $A$. By (2), $D\psi(D^*) \subseteq D\varphi(D^*)$. Thus

$$\varphi(D) = \{\delta g\overline{a} \mid \delta \in (C, D)\} \subseteq \{\gamma f\overline{a} \mid \gamma \in (B, D)\} = \psi(D).$$

So, for all $\delta \in (C, D)$, there is a $\gamma \in (B, D)$ such that $\delta g\overline{a} = \gamma f\overline{a}$. Since $\overline{a}$ generates $A$, $\delta g\overline{a} = \gamma f\overline{a}$ implies $\delta g = \gamma f$. Thus (1) holds.

□

As we have already noted $(C, \overline{c})$ freely realises $\varphi$ if and only if the map $(\overline{c}, -) : (C, -) \to (R^n, -)$ has image $F_\varphi$. So the following lemma replaces the use of free realisations.

**Lemma 3.4.** *Let $R$ be a ring. Let $f : A \to B$ be a morphism in* mod-$R$. *If $D \in R$*-mod *and $\gamma : (D, -) \to \ker(f \otimes -)$ is an epimorphism then for all $g : A \to C$, we have that* $\ker(f \otimes -) \subseteq \ker(g \otimes -)$ *if and only if* $\ker(f \otimes D) \subseteq \ker(g \otimes D)$.

*Proof.* Let $i : \ker(g \otimes -) \to A \otimes -$ and $j : \ker(f \otimes -) \to A \otimes -$ be the kernels of $g \otimes -$ and $f \otimes -$ respectively. Since $\gamma$ is an epimorphism and both $i$ and $j$ are monic, $j\gamma$ factors through $i$ if and only if $\ker(f \otimes -) \subseteq \ker(g \otimes -)$.

We now show that $\ker(f \otimes D) \subseteq \ker(g \otimes D)$ implies $j \circ \gamma$ factors through $i$. Since $\ker(f \otimes D) \subseteq \ker(g \otimes D)$, there is a $c \in \ker(g \otimes D)$ such that $i_D(c) = (j\gamma)_D(1_D)$. Let $\pi \in ((D, -), \ker(g \otimes -))$ be such that $\pi_D(1_D) = c$. Now $(i\pi)_D(1_D) = (j\gamma)_D(1_D)$. So $i\pi = j\gamma$ as required. □

The following lemma is a generalisation of (1) ⇒ (2) in 2.4.

**Lemma 3.5.** *Let $R$ be an Artin algebra over a commutative Artinian ring $S$. If $D^*$ is a left determiner for $f : A \to B \in$ mod-$R$ then there exists an $n \in \mathbb{N}$ and an epimorphism $\gamma : (D^n, -) \to \ker(f \otimes -)$.*

*Proof.* For any ring $R$, if $A \in$ mod-$R$ then all finitely presented subobjects of $A \otimes -$ are of the form $\ker(g \otimes -)$ for some $g : A \to B \in$ mod-$R$. This follows from the fact that $(R\text{-mod}, \mathrm{Ab})^{\mathrm{fp}}$ has enough injectives and that all injectives are of the form $B \otimes -$ for some $B \in$ mod-$R$ (see [GJ81, 5.5], [Pre09, 12.1.13]). Hence, all finitely presented subobjects of $A \otimes -$ are of the form $\ker \nu$ for some natural transformation $\nu : A \otimes - \to B \otimes -$. Further, since $h : A \otimes - \to B \otimes -$ is the zero morphism if and only if $h_R = 0$, all natural transformations $\nu : A \otimes - \to B \otimes -$ are of the form $g \otimes -$ for some $g : A \to B$.

Suppose that $D^*$ is a left determiner for $f : A \to B \in$ mod-$R$. Combining 3.2 and 3.3, we have that for all $g : A \to C \in$ mod-$R$, $\ker(f \otimes -) \subseteq \ker(g \otimes -)$

if and only if $\ker(f \otimes D) \subseteq \ker(g \otimes D)$. Thus if $F$ is any finitely presented subfunctor of $A \otimes -$, then $\ker(f \otimes -) \subseteq F$ if and only if $\ker(f \otimes D) \subseteq FD$. Since $A, B \in \text{mod-}R$ and $D \in R\text{-mod}$, $\ker(f \otimes D)$ is finitely generated as a module over $S$.

Let $a_1, ..., a_n$ generate $\ker(f \otimes D)$ over $S$. For each $1 \leq i \leq n$, let $\gamma_i : (D, -) \to \ker(f \otimes -)$ be such that $(\gamma_i)_D(1_D) = a_i$. Let $\gamma : (D^n, -) \to \ker(f \otimes -)$ be $(\gamma_1, ..., \gamma_n)$. Now the image of $\gamma_D$ is the whole of $\ker(f \otimes D)$. Thus the image of $\gamma$ is $\ker(f \otimes -)$ as required. □

Finally we are ready to prove 3.1.

*proof of 3.1.* Lemma 3.4 shows that any projective $(D, -)$ which maps epimorphically onto $\ker(f \otimes -)$ has the property that $\ker(f \otimes -) \subseteq \ker(g \otimes -)$ if and only if $\ker(f \otimes D) \subseteq \ker(g \otimes D)$. Lemma 3.3 says that this is true if and only if $\text{im}(g, D^*) \subseteq \text{im}(f, D^*)$. Lemma 3.2 says that $\ker(f \otimes -) \subseteq \ker(g \otimes -)$ if and only if $\text{im}(g, -) \subseteq \text{im}(f, -)$. So $D^*$ is a left determiner for $f$.

Suppose $C^*$ is a left determiner for $f$. Then, by 3.5, there is an $n \in \mathbb{N}$ and epimorphism $\gamma : (C^n, -) \to \ker(f \otimes -)$. Since $(D, -)$ is a projective cover for $\ker(f \otimes -)$, using general properties of projective covers, we have that $(D, -)$ is a direct summand of $(C^n, -)$. Therefore $D$ is a direct summand of $C^n$. □

**Corollary 3.6.** *Let $R$ be an Artin algebra and $f : A \to B \in \text{mod-}R$. If $- \otimes D$ is an injective hull for coker$((f, -))$ then $D^*$ is a left determiner for $f$ and if $C$ is a left determiner for $f$ then $D^*$ is a direct summand of $C^n$ for some $n \in \mathbb{N}$.*

*Proof.* Apply the Auslander-Gruson-Jensen dual $d$ as in [Pre09, Section 10.3] to 3.1. □

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University of Camerino, School of Science and Technologies, Division of Mathematics, Via Madonna delle Carceri 9, 62032 Camerino, Italy

*Email address*: `lorna.gregory@gmail.com`